QUANTUM GEOMETRY AND ITS APPLICATIONS

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Related Articles

Loop quantum gravity, Quantum dynamics in loop quantum gravity, quantum cosmology, black hole mechanics, quantum field theory in curved space-times, spin foams, canonical approach.

1 Introduction

In general relativity the gravitational field is encoded in the Riemannian geometry of space-time. Much of the conceptual compactness and mathematical elegance of the theory can be traced back to this central idea. The encoding is also directly responsible for the most dramatic ramifications of the theory: the big-bang, black holes and gravitational waves. However, it also leads one to the conclusion that space-time itself must end and physics must come to a halt at the big-bang and inside black holes, where the gravitational field becomes singular. But this reasoning ignores quantum physics entirely. When the curvature becomes large, of the order of $1/\ell_{\rm Pl}^2 = c^3/G\hbar$, quantum effects dominate and predictions of general relativity can no longer be trusted. In this 'Planck regime', one must use an appropriate synthesis of general relativity and quantum physics, i.e., a quantum gravity theory. The predictions of this theory are likely to be quite different from those of general relativity. In the real, quantum world, evolution may be completely non-singular. Physics may not come to a halt and quantum theory could extend classical space-time.

There are a number of different approaches to quantum gravity. One natural avenue is to retain the interplay between gravity and geometry but now use *quantum* Riemannian geometry in place of the standard, classical one. This is the key idea underlying loop quantum gravity. There are several calculations which indicate that the well-known failure of the standard perturbative approach to quantum gravity may be primarily due to its basic assumption that space-time can be modelled as a smooth continuum at *all* scales. In loop quantum gravity, one adopts a non-perturbative approach. There is no smooth metric in the background. Geometry is not only dynamical but quantum mechanical from 'birth'. Its fundamental excitations turn out to be 1-dimensional and polymer-like. The smooth continuum is only a coarse grained approximation. While a fully satisfactory quantum gravity theory still awaits us (in any approach), detailed investigations have been carried out to completion in simplified models —called mini and midi-superspaces. They show that quantum space-time does not end at singularities. Rather, quantum geometry serves as a 'bridge' to another large classical space-time.

This summary will focus on structural issues from a mathematical physics perspective. Complementary perspectives and further details can be found in articles on loop quantum gravity, canonical formalism, quantum cosmology, black hole thermodynamics and spin foams.

2 Basic Framework

The starting point is a Hamiltonian formulation of general relativity based on spin connections (Ashtekar, 1987). Here, the phase space $\mathbf{\Gamma}$ consists of canonically conjugate pairs $(A, \mathbb{P})$, where A is a connection on a 3-manifold M and $\mathbb{P}$ a 2-form, both of which take values in the Lie-algebra su(2). Since $\mathbf{\Gamma}$ can also be thought of as the phase space of the SU(2) Yang-Mills theory, in this approach there is a unified kinematic framework for general relativity which describes gravity and gauge theories which describe the other three basic forces of Nature. The connection A enables one to parallel transport chiral spinors (such as the left handed fermions of the standard electro-weak model) along curves in M. Its curvature is directly related to the electric and magnetic parts of the space-time *Riemann tensor*. The dual P of $\mathbb{P}$ plays a double role.[1] Being the momentum canonically conjugate to A, it is analogous to the Yang-Mills electric field. But (apart from a constant) it is also an orthonormal triad (with density weight 1) on M and therefore determines the positive definite ('spatial') 3-metric, and hence the Riemannian geometry of M. This dual role of P is a reflection of the fact that now SU(2) is the (double cover of the) group of rotations of the orthonormal spatial triads on M itself rather than of rotations in an 'internal' space associated with M.

To pass to quantum theory, one first constructs an algebra of 'elementary' func-

[1]The dual is defined via $\int_M \mathbb{P} \wedge \omega = \int_M P \lrcorner \omega$ for any 1-form ω on M.

tions on $\mathbf{\Gamma}$ (analogous to the phase space functions x and p in the case of a particle) which are to have unambiguous operator analogs. The holonomies

$$h_e(A) := \mathcal{P} \exp - \int_e A \tag{2.1}$$

associated with a curve/edge e on M is a (SU(2)-valued) configuration function on $\mathbf{\Gamma}$. Similarly, given a 2-surface S on M, and a su(2)-valued (test) function f on M,

$$P_{S,f} := \int_S \mathrm{Tr}\,(f\mathbb{P}) \tag{2.2}$$

is a momentum-function on $\mathbf{\Gamma}$, where Tr is over the su(2) indices.[2] The symplectic structure on $\mathbf{\Gamma}$ enables one to calculate the Poisson brackets $\{h_e\,, P_{S,f}\}$. The result is a linear combination of holonomies and can be written as a Lie derivative,

$$\{h_e,\, P_{S,f}\} = \mathcal{L}_{X_{S,f}}\, h_e\,, \tag{2.3}$$

where $X_{S,f}$ is a derivation on the ring generated by holonomy functions, and can therefore be regarded as a vector field on the configuration space $\mathcal{A}$ of connections. This is a familiar situation in classical mechanics of systems whose configuration space is a finite dimensional manifold. Functions h_e and vector fields $X_{S,f}$ generate a Lie algebra. As in quantum mechanics on manifolds, the first step is to promote this algebra to a quantum algebra by demanding that the commutator be given by $i\hbar$ times the Lie bracket. The result is a $\star$-algebra $\mathfrak{a}$, analogous to the algebra generated by operators $\exp i\lambda\hat{x}$ and $\hat{p}$ in quantum mechanics. By exponentiating the momentum operators $\hat{P}_{S,f}$ one obtains $\mathfrak{W}$, the analog of the quantum mechanical Weyl algebra generated by $\exp i\lambda\hat{x}$ and $\exp i\mu\hat{p}$.

The main task is to obtain the appropriate representation of these algebras. In that representation, *quantum* Riemannian geometry can be probed through the momentum operators $\hat{P}_{S,f}$, which stem from classical orthonormal triads. As in quantum mechanics on manifolds or simple field theories in flat space, it is convenient to divide the task into two parts. In the first, one focuses on the algebra $\mathfrak{C}$ generated by the configuration operators $\hat{h}_c$ and finds all its representations, and in the second one considers the momentum operators $\hat{P}_{S,f}$ to restrict the freedom.

$\mathfrak{C}$ is called the holonomy algebra. It is naturally endowed with the structure of an Abelian $C^\star$ algebra (with identity), whence one can apply the powerful machinery made available by the Gel'fand theory. This theory tells us that $\mathfrak{C}$ determines a unique compact, Hausdorff space $\bar{\mathcal{A}}$ such that the $C^\star$ algebra of all continuous functions on $\mathcal{A}$ is naturally isomorphic to $\mathfrak{C}$. $\bar{\mathcal{A}}$ is called the Gel'fand spectrum of

[2] For simplicity of presentation all fields are assumed to be smooth and curves/edges e and surfaces S, finite and piecewise analytic in a specific sense. The extension to smooth curves and surfaces was carried out by Baez and Sawin, Lewandowski and Thiemann, and Fleischhack. It is technically more involved but the final results are qualitatively the same.

$\mathfrak{C}$. It has been shown to consist of 'generalized connections' $\bar{A}$ defined as follows: $\bar{A}$ assigns to any oriented edge e in M an element $\bar{A}(e)$ of SU(2) (a 'holonomy') such that $\bar{A}(e^{-1}) = [\bar{A}(e)]^{-1}$; and, if the end point of e_1 is the starting point of e_2, then $\bar{A}(e_1 \circ e_2) = \bar{A}(e_1) \cdot \bar{A}(e_2)$. Clearly, every smooth connection A is a generalized connection. In fact, the space $\mathcal{A}$ of smooth connections has been shown to be dense in $\bar{\mathcal{A}}$ (with respect to the natural Gel'fand topology thereon). But $\bar{\mathcal{A}}$ has many more 'distributional elements'. The Gel'fand theory guarantees that every representation of the $C^\star$ algebra $\mathfrak{C}$ is a direct sum of representations of the following type: The underlying Hilbert space is $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu)$ for some measure μ on $\bar{\mathcal{A}}$ and (regarded as functions on $\bar{\mathcal{A}}$) elements of $\mathfrak{C}$ act by multiplication. Since there are many inequivalent measures on $\bar{\mathcal{A}}$, there is a multitude of representations of $\mathfrak{C}$. A key question is how many of them can be extended to representations of the full algebra $\mathfrak{a}$ (or $\mathfrak{W}$) without having to introduce any 'background fields' which would compromise diffeomorphism covariance. Quite surprisingly, the requirement that the representation be cyclic with respect to a state which is invariant under the action of the (appropriately defined) group Diff M of piecewise analytic diffeomorphisms on M singles out a *unique* irreducible representation. This result was established for $\mathfrak{a}$ by Lewandowski, Okołów, Sahlmann and Thiemann, and for $\mathfrak{W}$ by Fleischhack. It is the quantum geometry analog to the seminal results by Segal and others that characterized the Fock vacuum in Minkowskian field theories. However, while that result assumes not only Poincaré invariance but also specific (namely free) dynamics, it is striking that the present uniqueness theorems make no such restriction on dynamics. The requirement of diffeomorphism invariance is surprisingly strong and makes the 'background independent' quantum geometry framework surprisingly tight.

This representation had been constructed by Ashtekar, Baez and Lewandowski some ten years before its uniqueness was established. The underlying Hilbert space is given by $\mathcal{H} = L^2(\bar{\mathcal{A}}, d\mu_o)$ where μ_o is a diffeomorphism invariant, faithful, regular Borel measure on $\bar{\mathcal{A}}$, constructed from the normalized Haar measure on SU(2). Typical quantum states can be visualized as follows. Fix: (i) a graph[3] α on M, and, (ii) a smooth function ψ on $[\mathrm{SU}(2)]^n$. Then, the function

$$\Psi_\gamma(\bar{A}) := \psi(\bar{A}(e_1), \dots \bar{A}(e_n)) \tag{2.4}$$

on $\bar{\mathcal{A}}$ is an element of $\mathcal{H}$. Such states are said to be *cylindrical* with respect to the graph α and their space is denoted by Cyl_α. These are 'typical states' in the sense that $\mathrm{Cyl} := \cup_\alpha \mathrm{Cyl}_\alpha$ is dense in $\mathcal{H}$. Finally, as ensured by the Gel'fand theory, the holonomy (or configuration) operators $\hat{h}_e$ act just by multiplication. The momentum operators $\hat{P}_{S,f}$ act as Lie-derivatives: $\hat{P}_{S,f}\,\Psi = -i\hbar\,\mathcal{L}_{X_{S,f}}\,\Psi$.

Remark: Given any graph α in M, and a labelling of each of its edges by a non-trivial irreducible representation of SU(2) (i.e., by a non-zero half integer j), one can construct a *finite* dimensional Hilbert space $\mathcal{H}_{\alpha,\vec{j}}$ which can be thought of as the state

[3] By a graph on M we mean a set of a finite number of embedded, oriented intervals called edges. If two edges intersect, they do so only at one or both ends, called vertices.

space of a spin system 'living on' the graph α. The full Hilbert space admits a simple decomposition: $\mathcal{H} = \oplus_{\alpha,\vec{j}} \mathcal{H}_{\alpha,\vec{j}}$. This is called the spin-network decomposition. The geometric operators discussed in the next section leave each $\mathcal{H}_{\alpha,\vec{j}}$ invariant. Therefore, the availability of this decomposition greatly simplifies the task of analyzing their properties.

3 Geometric Operators

In the classical theory, $E := 8\pi G\gamma P$ has the interpretation of an orthonormal triad field (or a 'moving frame') on M (with density weight 1). Here γ is a dimensionless, strictly positive number, called the Barbero-Immirzi parameter, which arises as follows. Because of emphasis on connections, in the classical theory the first order Palatini action is a more natural starting point than the second order Einstein-Hilbert action. Now, there is a freedom to add a term to the Palatini action which vanishes when Bianchi identities are satisfied and therefore does not change the equations of motion. γ arises as the coefficient of this term. In some respects γ is analogous to the θ parameter of Yang-Mills theory. Indeed, while theories corresponding to any permissible values of γ are related by a canonical transformation classically, quantum mechanically this transformation is not unitarily implementable. Therefore, although there is a unique representation of the algebra $\mathfrak{a}$ (or $\mathfrak{W}$) there is a 1-parameter family of inequivalent representations of the algebra of *geometric operators* generated by suitable functions of orthonormal triads E, each labelled by the value of γ. This is a genuine quantization ambiguity. As with the θ ambiguity in QCD, the actual value of γ in Nature has to be determined experimentally. The current strategy in quantum geometry is to fix its value through a thought experiment involving black hole thermodynamics (see below).

The basic object in quantum Riemannian geometry is the triad flux operator $\hat{E}_{S,f} := 8\pi G\gamma\, \hat{P}_{S,f}$. It is self-adjoint and all its eigenvalues are *discrete*. To define other geometric operators such as the area operator $\hat{A}_S$ associated with a surface S or a volume operator $\hat{V}_R$ associated with a region R, one first expresses the corresponding phase space functions in terms of the 'elementary' functions E_{S_i,f_i} using suitable surfaces S_i and test functions f_i and then promotes E_{S_i,f_i} to operators. Even though the classical expressions are typically non-polynomial functions of E_{S_i,f_i}, the final operators are all well-defined, self-adjoint and *with purely discrete eigenvalues.* Therefore, in the sense of the word used in elementary quantum mechanics (e.g. of the hydrogen atom), one says that geometry is *quantized.* Because the theory has no background metric or indeed any other background field, all geometric operators transform covariantly under the action of the Diff M. This diffeomorphism covariance makes the final expressions of operators rather simple. In the case of the area operator, for example, the action of $\hat{A}_S$ on a state Ψ_α (2.4) depends entirely on the points of intersection of the surface S and the graph α and involves only right and left invariant vector fields on copies of SU(2) associated with edges of α which intersect S. In the case of the volume operator $\hat{V}_R$, the action depends on the vertices of α contained in R and, at each vertex, involves the right and left invariant vector fields on copies of SU(2) associated with edges that meet at each vertex.

To display the explicit expressions of these operators, let us first define on Cyl_α three basic operators $\hat{J}_j^{(v,e)}$, with $j \in \{1,2,3\}$, associated with the pair consisting of an edge e of α and a vertex v of e:

$$\hat{J}_j^{(v,e)}\Psi_\alpha(\bar{A}) \;=\; \begin{cases} i\frac{d}{dt}|_{t=0}\psi_\alpha(\ldots,\, U_e(\bar{A})\exp(t\tau_j),\, \ldots) & \text{if } e \text{ begins at } v \\ i\frac{d}{dt}|_{t=0}\psi_\alpha(\ldots,\, \exp(-t\tau_j)U_e(\bar{A}),\, \ldots) & \text{if } e \text{ ends at } v, \end{cases} \tag{3.5}$$

where τ_j denotes a basis in su(2) and '$\ldots$' stands for the rest of the arguments of Ψ_α which remain unaffected. The quantum area operator A_s is assigned to a finite 2-dimensional sub-manifold S in M. Given a cylindrical state we can always represent it in the form (2.4) using a graph α adapted to S, such that every edge e either intersects S at exactly one end point, or is contained in the closure $\bar{S}$, or does not intersect $\bar{S}$. For each vertex v of the graph α which lies on S, the family of edges intersecting v can be divided into 3 classes: edges $\{e_1, ..., e_u\}$ lying on one side (say 'above') S, edges $\{e_{u+1}, ..., e_{u+d}\}$ lying on the other side (say 'below') and edges contained in S. To each v we assign a generalized Laplace operator

$$\Delta_{S,v} = -\eta^{ij}\left(\sum_{I=1}^{u}\hat{J}_i^{(v,e_I)} - \sum_{I=u+1}^{u+d}\hat{J}_i^{(v,e_I)}\right)\left(\sum_{K=1}^{u}\hat{J}_j^{(v,e_K)} - \sum_{K=u+1}^{u+d}\hat{J}_j^{(v,e_K)}\right), \tag{3.6}$$

where η_{ij} stands for $-\frac{1}{2}$ the Killing form on su(2). Now, the action of the quantum area operator $\hat{A}_S$ on Ψ_α is defined as follows

$$\hat{A}_S\Psi_\alpha \;=\; 4\pi\gamma\ell_{\mathrm{Pl}}^2\sum_{v\in S}\sqrt{-\Delta_{S,v}}\,\Psi_\alpha. \tag{3.7}$$

The quantum area operator has played the most important role in applications. Its complete spectrum is known in a closed form. Consider arbitrary sets $j_I^{(u)}, j_I^{(d)}$ and $j_I^{(u+d)}$ of half integers, subject to the condition

$$j_I^{(u+d)} \;\in \{|j_I^{(u)} - j_I^{(d)}|, |j_I^{(u)} - j_I^{(d)}| + 1, ..., j_I^{(u)} + j_I^{(d)}\}\,, \tag{3.8}$$

where I runs over any finite number of integers. The general eigenvalues of the area operator are given by:

$$a_S \;=\; 4\pi\gamma\ell_{\mathrm{Pl}}^2\sum_I\sqrt{2j_I^{(u)}(j_I^{(u)}+1) + 2j_I^{(d)}(j_I^{(d)}+1) - j_I^{(u+d)}(j_I^{(u+d)}+1)}\,. \tag{3.9}$$

On the physically interesting sector of SU(2)-gauge invariant subspace $\mathcal{H}_{\mathrm{inv}}$ of $\mathcal{H}$, the lowest eigenvalue of $\hat{A}_S$ —*the area gap*— depends on some global properties of

S. Specifically, it 'knows' whether the surface is open, or a 2-sphere, or, if M is a 3-torus, a (non-trivial) 2-torus in M. Finally, on $\mathcal{H}_{\rm inv}$, one is often interested only in the subspace of states Ψ_α where α has no edges which lie within a given surface S. Then, the expression of eigenvalues simplifies considerably:

$$a_S \;=\; 8\pi\gamma\ell_{\rm Pl}^2 \sum_I \sqrt{j_I(j_I+1)}\,. \tag{3.10}$$

To display the action of the quantum volume operator $\hat{V}_R$, for each vertex v of a given graph α, let us first define an operator $\hat{q}_v$ on Cyl_α.

$$\hat{q}_v \;=\; (8\pi\gamma\ell_{\rm Pl}^2)^3 \frac{1}{48} \sum_{e,e',e''} \epsilon(e,e',e'') c^{ijk} \hat{J}_i^{(v,e)} \hat{J}_j^{(v,e')} \hat{J}_k^{(v,e'')}, \tag{3.11}$$

where e, e' and e'' run over the set of edges intersecting v, $\epsilon(e,e',e'')$ takes values ± 1 or 0 depending on the orientation of the half-lines tangent to the edges at v, $[\tau_i,\tau_j] = c^k{}_{ij}\tau_k$ and the indices are raised by the tensor η_{ij}. The action of the quantum volume operator on a cylindrical state (2.4) is then given by

$$\hat{V}_R \Psi_\alpha \;=\; \kappa_o \sum_{v\in R} \sqrt{|\hat{q}_v|}.\Psi_\alpha, \tag{3.12}$$

Here κ_o is an overall constant, independent of a graph constant resulting from an averaging.

The volume operator plays an unexpectedly important role in the definition of both the gravitational and matter contributions to the scalar constraint operator which dictates dynamics. Finally, a notable property of the volume operator is the following. Let $R(p,\epsilon)$ be a family of neighborhoods of a point $p \in M$. Then, as indicated above, $\hat{V}_{R(p,\epsilon)}\Psi_\alpha = 0$ if α has no vertex in the neighborhood. However, if α has a vertex at p

$$\lim_{\epsilon\to 0} \hat{V}_{R(x,\epsilon)}\, \Psi_\alpha$$

exists but is *not necessarily zero.* This is a reflection of the 'distributional' nature of quantum geometry.

Remark: States $\Psi_\alpha \in \mathrm{Cyl}$ have support only on the graph α. In particular, they are simply annihilated by geometric operators such as $\hat{A}_S$ and $\hat{V}_R$ if the support of the surface S and the region R does not intersect the support of α. In this sense the fundamental excitations of geometry are 1-dimensional and geometry is 'polymer-like'. States Ψ_α where α is just a 'small graph' are highly quantum mechanical —like states in QED representing just a few photons. Just as coherent states in QED require an infinite superposition of such highly quantum states, to obtain a semi-classical state approximating a given classical geometry, one has to superpose a very large number of such elementary states. More precisely, in the Gel'fand triplet $\mathrm{Cyl} \subset \mathcal{H} \subset \mathrm{Cyl}^\star$, semi-classical states belong to the dual $\mathrm{Cyl}^\star$ of Cyl.

4 Applications

Since quantum Riemannian geometry underlies loop quantum gravity and spin-foam models, all results obtained in these frameworks can be regarded as its applications. Among these, there are two which have led to resolutions of long standing issues. The first concerns black hole entropy, and the second, quantum nature of the big-bang.

4.1 Black holes

Seminal advances in fundamentals of black hole physics in the mid seventies suggested that the entropy of large black holes is given by $S_{\rm BH} = (a_{\rm hor}/4\ell_{\rm Pl}^2)$, where $a_{\rm hor}$ is the horizon area. This immediately raised a challenge to potential quantum gravity theories: Give a statistical mechanical derivation of this relation. For familiar thermodynamic systems, a statistical mechanical derivation begins with an identification the microscopic degrees of freedom. For a classical gas, these are carried by molecules; for the black body radiation, by photons and for a ferromagnet, by Heisenberg spins. What about black holes? The microscopic building blocks can not be gravitons because the discussion involves stationary black holes. Furthermore the number of microscopic states is absolutely huge: some $\exp 10^{77}$ for a solar mass black hole, a number that completely dwarfs the number of states of systems one normally encounters in statistical mechanics. Where does this huge number come from? In loop quantum gravity, this is the number of states of the *quantum horizon geometry.*

The idea behind the calculation can be heuristically explained using the *It from Bit* argument, put forward by Wheeler in the nineties. Divide the black hole horizon in to elementary cells, each with one Planck unit of area, $\ell_{\rm Pl}^2$ and assign to each cell two microstates. Then the total number of states $\mathcal{N}$ is given by $\mathcal{N} = 2^n$ where $n = (a_{\rm hor}/\ell_{\rm Pl}^2)$ is the number of elementary cells, whence entropy is given by $S = \ln \mathcal{N} \sim a_{\rm hor}$. Thus, apart from a numerical coefficient, the entropy ('*It*') is accounted for by assigning two states ('*Bit*') to each elementary cell. This qualitative picture is simple and attractive. However, the detailed derivation in quantum geometry has several new features.

First, Wheeler's argument would apply to any 2-surface, while in quantum geometry the surface must represent a horizon in equilibrium. This requirement is encoded in a certain boundary condition that the canonically conjugate pair $(A, \mathbb{P})$ must satisfy at the surface and plays a crucial role in the quantum theory. Second, the area of each elementary cell is not a fixed multiple of $\ell_{\rm Pl}^2$ but is given by (3.10), where I labels the elementary cells and j_I can be any half integers (such that the sum is within a small neighborhood of the classical area of the black hole under consideration). Finally, the number of quantum states associated with an elementary cell labelled by j_I is not 2 but $(2j_I + 1)$.

The detailed theory of the quantum horizon geometry and the standard statistical mechanical reasoning is then used to calculate the entropy and the temperature. For large black holes, the leading contribution to entropy is proportional to the horizon area, in agreement with quantum field theory in curved space-times.[4] However, as one would expect, the proportionality factor depends on the Barbero-Immirzi parameter γ and so far loop quantum gravity does not have an independent way to determine its value. The current strategy is to determine γ by *requiring* that, for the Schwarzschild black hole, the leading term agree *exactly* with Hawing's semi-classical answer. This requirement implies that γ is the root of algebraic equation and its value is given by $\gamma \approx 0.2735$. Now quantum geometry theory is completely fixed. One can calculate entropy of other black holes, with angular momentum and distortion. A non-trivial check on the strategy is that for all these cases, the coefficient in the leading order term again agrees with Hawking's semi-classical result.

The detailed analysis involves a number of structures of interest to mathematical physics. First, the intrinsic horizon geometry is described by an U(1) Chern-Simons theory on a punctured 2-sphere (the horizon), the level k of the theory being given by $k = a_{\rm hor}/4\pi\gamma\,\ell_{\rm Pl}^2$. The punctures are simply the intersections of the excitations of the polymer geometry in the bulk with the horizon 2-surface. Second, because of the horizon boundary conditions, in the classical theory the gauge group SU(2) is reduced to U(1) at the horizon. At each puncture, it is further reduced to the discrete subgroup $\mathbb{Z}_k$ of U(1), —sometimes referred to as a 'quantum U(1) group'. Third, the 'surface phase space' associated with the horizon is represented by a non-commutative torus. Finally, the surface Chern-Simons theory is entirely unrelated to the bulk quantum geometry theory but the quantum horizon boundary condition requires that the spectrum of a certain operator in the Chern-Simons theory must be identical to that of another operator in the bulk theory. The surprising fact is that there is an exact agreement. Without this seamless matching, a coherent descriptions of the quantum horizon geometry would not have been possible.

The main weakness of this approach to black hole entropy stems from the Barbero-Immirzi ambiguity. The argument would be much more compelling if the value of γ were determined by independent considerations, without reference to black hole entropy.[5] It's primary strengths are two folds. First, the calculation encompasses all realistic black holes —not just extremal or near-extremal— including the astrophysical ones, which may be highly distorted. Hairy black holes of mathematical physics and cosmological horizons are also encompassed. Second, in contrast to other approaches, one works directly with the physical, curved geometry around black holes rather than with a flat space system which has the same number of states as the black hole of interest.

[4]The sub-leading term $-\frac{1}{2}\ln\left(a_{\rm hor}/\ell_{\rm Pl}^2\right)$ is a quantum gravity correction to Hawking's semi-classical result. This correction, with the $-1/2$ factor, is robust in the sense that it also arises in other approaches.

[5]By contrast, for *extremal* black holes, string theory provides the correct coefficient without any adjustable parameter. The AdS/CFT duality hypothesis (as well as another semi-quantitative) arguments have been used to encompass certain black holes which are away from extremality. But in these cases, it is not known if the numerical coefficient is 1/4 as in Hawking's analysis.

4.2 The big bang

Most of the work in physical cosmology is carried out using spatially homogeneous and isotropic models and perturbations thereon. Therefore, to explore the quantum nature of the big-bang, it is natural to begin by assuming these symmetries. Then the space-time metric is determined simply by the scale factor $a(t)$ and matter fields $\phi(t)$ which depend only on time. Thus, because of symmetries, one is left with only a finite number of degrees of freedom. Therefore, field theoretic difficulties are by-passed and passage to quantum theory is simplified. This strategy was introduced already in the late sixties and early seventies by DeWitt and Misner. Quantum Einstein's equations now reduce to a single differential equation of the type

$$\frac{\partial^2}{\partial a^2}(f(a)\Psi(a,\phi)) = \mathrm{const}\, \hat{H}_\phi\, \Psi(a,\phi) \tag{4.13}$$

on the wave function $\Psi(a,\phi)$, where $\hat{H}_\phi$ is the matter Hamiltonian and $f(a)$ reflects the freedom in factor ordering. Since the scale factor a vanishes at the big-bang, one has to analyze the equation and its solutions near $a = 0$. Unfortunately, because of the standard form of the matter Hamiltonian, coefficients in the equation diverge at $a = 0$ and the evolution can not be continued across the singularity unless one introduces unphysical matter or a new principle. A well-known example of new input is the Hartle-Hawking boundary condition which posits that the universe starts out without any boundary and a metric with positive definite signature and later makes a transition to a Lorentzian metric.

Bojowald and others have shown that the situation is quite different in loop quantum cosmology because quantum geometry effects make a qualitative difference near the big-bang. As in older quantum cosmologies, one carries out a symmetry reduction at the classical level. The final result differs from older theories only in minor ways. In the homogeneous, isotropic case, the freedom in the choice of the connection is encoded in a single function $c(t)$ and, in that of the momentum/triad, in another function $p(t)$. The scale factor is given by $a^2 = |p|$. (The variable p itself can assume both signs; positive if the triad is left handed and negative if it is right handed. p vanishes at degenerate triads which are permissible in this approach.) The system again has only a finite number of degrees of freedom. However, quantum theory turns out to be *inequivalent* to that used in older quantum cosmologies.

This surprising result comes about as follows. Recall that in quantum geometry, one has well-defined holonomy operators $\hat{h}$ but there is no operator corresponding to the connection itself. In quantum mechanics, the analog would be for operators $\hat{U}(\lambda)$ corresponding to the classical functions $\exp i\lambda x$ to exist but not be weakly continuous in λ; the operator $\hat{x}$ would then not exist. Once the requirement of weak continuity is dropped, von Neumann's uniqueness theorem no longer holds and the Weyl algebra can have inequivalent irreducible representations. The one used in loop quantum cosmology is the direct analog of full quantum geometry. While the space $\mathcal{A}$ of smooth connections reduces just to the real line $\mathbb{R}$, the space $\bar{\mathcal{A}}$ of generalized connections reduces to the Bohr compactification $\bar{\mathbb{R}}_{\rm Bohr}$ of the real line. (This space was introduced by the mathematician Harold Bohr (Nils' brother) in his theory of almost periodic functions. It arises in the present application because holonomies turn out to be almost periodic functions of c.) The Hilbert space of states is thus $\mathcal{H} = L^2(\bar{\mathbb{R}}_{\rm Bohr}, d\mu_o)$ where μ_o is the Haar measure on (the Abelian group) $\bar{\mathbb{R}}_{\rm Bohr}$. As in full quantum geometry, the holonomies act by multiplication and the triad/momentum operator $\hat{p}$ via Lie-derivatives.

To facilitate comparison with older quantum cosmologies, it is convenient to use a representation in which $\hat{p}$ is diagonal. Then, quantum states are functions $\Psi(p, \phi)$. But the Wheeler-DeWitt equation is now replaced by a *difference* equation:

$$C^+(p)\,\Psi(p + 4p_o,\, \phi) + C^o(p)\,\Psi(p, \phi) + C^-(p)\,\Psi(p - 4p_o)(\phi) = \text{const}\, \hat{H}_\phi \Psi(p, \phi) \quad (4.14)$$

where p_o is determined by the lowest eigenvalue of the area operator ('area gap') and the coefficients $C^\pm(p)$ and $C^o(p)$ are functions of p. In a backward 'evolution', given Ψ at $p+4$ and p, such a 'recursion relation' determines Ψ at $p-4$, provided C^- does not vanish at $p-4$. The coefficients are well-behaved and nowhere vanishing, whence the 'evolution' does not stop at any finite p, either in the past or in the future. Thus, near $p = 0$ this equation is drastically different from the Wheeler DeWitt equation (4.13). However, for large p —i.e., when the universe is large— it is well approximated by (4.13) and smooth solutions of (4.13) are approximate solutions of the fundamental discrete equation (4.14) in a precise sense.

To complete quantization, one has to introduce a suitable Hilbert space structure on the space of solutions to (4.14), identify physically interesting operators and analyze their properties. For simple matter fields, this program has been completed. With this machinery at hand one begins with semi-classical states which are peaked at configurations approximating the classical universe at late times (e.g., now) and evolves backwards. Numerical simulations show that the state remains peaked at the classical solution till *very* early times when the matter density becomes of the order of Planck density. This provides, in particular, a justification, from first principles, for the assumption that space-time can be taken to be classical even at the onset of the inflationary era, just a few Planck times after the (classical) big-bang. While one would expect a result along these lines to hold on physical grounds, technically it is non-trivial to obtain semi-classicality over such huge domains. However, in the Planck regime near the big-bang, there are major deviations from the classical behavior. Effectively, gravity becomes repulsive, the collapse is halted and then the universe re-expands. Thus, rather than modifying space-time structure just in a tiny region near the singularity, quantum geometry effects open a bridge to another large classical universe. These are dramatic modifications of the classical theory.

For over three decades, hopes have been expressed that quantum gravity would provide new insights into the true nature of the big-bang. Thanks to quantum geometry effects, these hopes have been realized and many of the long standing questions have been answered. While the final picture has some similarities with other approaches, (e.g., 'cyclic universes', or pre-big-bang cosmology), only in loop quantum cosmology is there a fully deterministic evolution across what was the classical big-bang. However, *so far detailed results have been obtained only in simple models.* The major open issue is the inclusion of perturbations and subsequent comparison with observations.

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