BLACK HOLE MECHANICS

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1 Introduction

Over the last thirty years, black holes have been shown to have a number of surprising properties. These discoveries have revealed unforeseen relations between the otherwise distinct areas of general relativity, quantum physics and statistical mechanics. This interplay, in turn, led to a number of deep puzzles at the very foundations of physics. Some have been resolved while others continue to baffle physicists. The starting point of these fascinating developments was the discovery of laws of black hole mechanics by Bardeen, Bekenstein, Carter and Hawking. They dictate the behavior of black holes in equilibrium, under small perturbations away from equilibrium, and in fully dynamical situations. While they are consequences of classical general relativity alone, they have a close similarity with the laws of thermodynamics. The origin of this seemingly strange coincidence lies in quantum physics. For further discussion, see articles on quantum field theory in curved space-times and quantum gravity.

The focus of this review is just on black hole mechanics. The discussion is divided into three parts. In the first, we will introduce the notions of event horizons and black hole regions and discuss properties of globally stationary black holes. In the second, we will consider black holes which are themselves in equilibrium but in surroundings which may be time-dependent. Finally, in the third part, we summarize what is known in the fully dynamical situations. For simplicity, all manifolds and field are assumed to be smooth and, unless otherwise stated, space-time is assumed to be 4-dimensional, with a metric of signature -,+,+,+, and the cosmological constant is assumed to be zero. An arrow under a space-time index denotes the pull-back of that index to the horizon.

2 Global equilibrium

To capture the intuitive notion that black hole is a region from which signals can not escape to the asymptotic part of space-time, one needs a precise definition of future infinity. The standard strategy is to use Penrose's conformal boundary $\mathcal{I}^+$. A *Black-hole region* $\mathcal{B}$ of a space-time $(\mathbb{M}, g_{ab})$ is defined as $\mathcal{B} = \mathbb{M} \setminus I^-(\mathcal{I}^+)$, where I^- denotes 'chronological past'. The boundary $\partial\mathcal{B}$ of the black hole region is called the *event horizon* and denoted by $\mathcal{E}$. Thus, $\mathcal{E}$ is the boundary of the past of $\mathcal{I}^+$. It therefore follows that $\mathcal{E}$ is a null 3-surface, ruled by future inextendible null geodesics without caustics. If the space-time is globally hyperbolic, an 'instant of time' is represented by a Cauchy surface M. The intersection of $\mathcal{B}$ with M may have several disjoint components, each representing a black hole at that instant of time. If M' is a Cauchy surface to the future of M, the number of disjoint components of $M' \cup \mathcal{B}$ in the causal future of $M \cup \mathcal{B}$ must be less than or equal to those of $M \cup \mathcal{B}$ (see Hawking & Ellis 1973). Thus, black holes can merge but can not bifurcate. (By a time reversal, i.e. by replacing $\mathcal{I}^+$ with $\mathcal{I}^-$ and I^- with I^+, one can define a white hole region $\mathcal{W}$. However, here we will focus only on black holes.)

A space-time $(\mathbb{M}, g_{ab})$ is said to be *stationary* (i.e., time-independent) if g_{ab} admits a Killing field t^a which represents an asymptotic time-translation. By convention, t^a is assumed to be unit at infinity. $(\mathbb{M}, g_{ab})$ is said to be axi-symmetric if g_{ab} admits a Killing field ϕ^a generating an $SO(2)$ isometry. By convention ϕ^a is normalized such that the affine length of its integral curves is 2π. Stationary space-times with non-trivial $\mathbb{M} \setminus I^-(\mathcal{I}^+)$ represent black holes which are in global equilibrium. In the Einstein-Maxwell theory in 4 dimensions, there exists a unique 3-parameter family of stationary black hole solutions, generally parameterized by mass m, angular momentum J and electric charge Q. This is the celebrated Kerr-Newman family. Therefore, in general relativity a great deal of work on black holes has focused on these solutions and perturbations thereof. The Kerr-Newman family is axi-symmetric and furthermore its metric has the property that the 2-flats spanned by the Killing fields t^a and ϕ^a are orthogonal to a family of 2-surfaces. This property is called '$t - \phi$ orthogonality'. Note however that uniqueness *fails* in higher dimensions, and also in presence of non-Abelian gauge fields or rings of perfect fluids around black holes in 4 dimensions. In mathematical physics, there is significant literature on the new stationary black hole solutions in Einstein-Yang-Mills-Higgs theories. These are called 'hairy black holes'. Research on stationary black hole solutions with rings received a boost by a recent discovery that these black holes can violate the Kerr inequality $J \leq Gm^2$ between angular momentum J and mass m.

A null 3-manifold $\mathcal{K}$ in $\mathbb{M}$ is said to be a *Killing horizon* if g_{ab} admits a Killing field K^a which is everywhere normal to $\mathcal{K}$. On a Killing horizon, one can show that the acceleration of K^a is proportional to K^a itself:

$$K^a \nabla_a K^b = \kappa K^b \,. \tag{2.1}$$

The proportionality function κ is called *surface gravity.* We will show in the next section that if a mild energy condition holds on $\mathcal{K}$ then κ must be constant. Note that if we rescale K^a via $K^a \to cK^a$, where c is a constant, surface gravity also rescales as $\kappa \to c\kappa$.

In the Kerr-Newman family, the event horizon is a Killing horizon. More generally, if an axi-symmetric, stationary black hole space-time $(\mathbb{M}, g_{ab})$ satisfies the '$t - \phi$ orthogonality' property, its event horizon $\mathcal{E}$ is a Killing horizon.[1] In these cases, the normalization freedom in K^a is fixed by requiring that K^a have the form

$$K^a = t^a + \Omega \phi^a \tag{2.2}$$

on the horizon, where Ω is a constant, called the *angular velocity of the horizon.* The resulting κ is called the surface gravity of the black hole. It is remarkable that κ is constant for all such black holes, even when their horizon is highly distorted (i.e. far from being spherically symmetric) either due to rotation or due to external matter fields. This is analogous to fact that the temperature of a thermodynamical system in equilibrium is constant, independently of the details of the system. In analogy with thermodynamics, constancy of κ is referred to as the *zeroth law of black hole mechanics.*

Next, let us consider an infinitesimal perturbation δ within the 3-parameter Kerr-Newman family. A simple calculation shows that the changes in the Arnowitt-Deser-Misner (ADM) mass m, angular momentum J, and the total charge Q of the space-time and in the area a of the horizon are constrained via

$$\delta m = \frac{\kappa}{8\pi G}\, \delta a + \Omega\, \delta J + \Phi \delta Q \tag{2.3}$$

where the coefficients κ, Ω, Φ are black hole parameters, $\Phi = A_a K^a$ being the electrostatic potential at the horizon. The last two terms, $\Omega \delta J$ and $\Phi \delta Q$, have the interpretation of 'work' required to spin the black hole up by an amount δJ or to increase its charge by δQ. Therefore (2.3) has a striking resemblance to the first law, $\delta E = T\delta S + \delta W$, of thermodynamics if (as the zeroth law suggests) κ is made proportional to the temperature T and the horizon area a, to the entropy S. Therefore, (2.3) and its generalizations discussed below are referred to as the *first law of black hole mechanics.*

In Kerr-Newman space-times, the only contribution to the stress-energy tensor comes from the Maxwell field. Bardeen, Carter and Hawking (1973) consider stationary black holes with matter such as perfect fluids in the exterior region and stationary

[1] Although one can envisage stationary black holes in which these additional symmetry conditions are not met, this possibility has been ignored in black hole mechanics on stationary space-times. Quasi-local horizons, discussed below, do not require any space-time symmetries.

perturbations δ thereof. Using Einstein's equations, they show that the form (2.3) of the first law does not change; the only modification is addition of certain matter terms on the right side which can be interpreted as the work δW done on the total system. A generalization in another direction was made by Iyer and Wald (1994) using Noether currents. They allow *non-stationary* perturbations and, more importantly, drop the restriction to general relativity. Instead, they consider a wide class of diffeomorphism invariant Lagrangian densities $L(g_{ab}, R_{abcd}, \nabla_a R_{bcde}, \ldots, \Phi^{\cdot\cdot}{}_{\cdot\cdot}, \nabla_a \Phi^{\cdot\cdot}{}_{\cdot\cdot} \ldots)$ which depend on the metric g_{ab}, matter fields $\Phi^{\cdot\cdot}{}_{\cdot\cdot}$ and a finite number of derivatives of the Riemann tensor and matter fields. Finally, they restrict themselves to $\kappa \neq 0$. In this case, on the maximal analytic extension of the space-time, the Killing field K^a vanishes on a 2-sphere S_o called the bifurcate horizon. Then, (2.3) is generalized to:

$$\delta M = \frac{\kappa}{2\pi} \delta S_{\rm hor} + \delta W \, . \tag{2.4}$$

Here δW again represents 'work terms' and $S_{\rm hor}$ is given by:

$$S_{\rm hor} = -2\pi \oint_{S_o} \frac{\delta L}{\delta R_{abcd}} \, n_{ab} n_{cd} \, , \tag{2.5}$$

where n_{ab} is the bi-normal to S_o (with $n_{ab}n^{ab} = -2$), and the functional derivative inside the integral is evaluated by formally viewing the Riemann tensor as a field independent of the metric. For the Einstein-Hilbert action, this yields $S_{\rm hor} = a/4G$ and one recovers (2.3).

These results are striking. However, the underlying assumptions have certain unsatisfactory aspects. First, although the laws are meant to refer just to black holes, one assumes that the entire space-time is stationary. In thermodynamics, by contrast, one only assumes that the system under consideration is in equilibrium, not the whole universe. Second, in the first law, quantities a, Ω, Φ are evaluated at the horizon while M, J are evaluated at infinity and include contributions from possible matter fields outside the black hole. A more satisfactory law of black hole mechanics would involve attributes of the black hole alone. Finally, the notion of the event horizon is extremely global and teleological since it explicitly refers to $\mathcal{I}^+$. An event horizon may well be developing in the very room you are sitting today in anticipation of a gravitational collapse in the center of our galaxy which may occur a billion years hence. This feature makes it impossible to generalize the first law to fully dynamical situations and relate the change in the event horizon area to the flux of energy and angular momentum falling across it. Indeed, one can construct explicit examples of dynamical black holes in which an event horizon $\mathcal{E}$ forms *and grows* in the flat part of a space-time where nothing happens physically. These considerations call for a replacement of $\mathcal{E}$ by a quasi-local horizon which leads to a first law involving only horizon attributes, and which can grow only in response to the influx of energy. These horizons are discussed in the next two sections.

3 Local equilibrium

The key idea here is drop the requirement that space-time should admit a stationary Killing field and ask only that the intrinsic horizon geometry be time-independent. Consider a null 3-surface Δ in a space-time $(\mathbb{M}, g_{ab})$ with a future pointing normal field ℓ^a. The pull-back $q_{ab} := \underleftarrow{g_{ab}}$ of the space-time metric to Δ is the intrinsic, degenerate 'metric' of Δ with signature 0,+,+. The first condition is that it be 'time-independent', i.e. $\mathcal{L}_\ell\, q_{ab} = 0$ on Δ. Then by restriction, the space-time ∇ induces a natural derivative operator D on Δ. While D is compatible with q_{ab}, i.e. $D_a q_{bc} = 0$, it is not uniquely determined by this property because q_{ab} is degenerate. Thus, D has extra information, not contained in q_{ab}. The pair (q_{ab}, D) is said to determine the *intrinsic geometry* of the null surface Δ. This notion leads to a natural notion of a horizon in local equilibrium. Let Δ be a null, 3-dimensional sub-manifold of $(\mathbb{M}, g_{ab})$ with topology $\mathbb{S} \times \mathbb{R}$, where $\mathbb{S}$ is compact and without boundary.

Definition 1: Δ is said to be *isolated horizon* if it admits a null normal ℓ^a such that:
i) $\mathcal{L}_\ell\, q_{ab} = 0$ and $[\mathcal{L}_\ell,\, D] = 0$ on Δ; and
ii) $-T^a{}_b \ell^b$ is a future pointing causal vector on Δ.
On can show that, generically, this null normal field ℓ^a is unique up to rescalings by positive *constants*.

Both conditions are local to Δ. In particular, $(\mathbb{M}, g_{ab})$ is not required to be asymptotically flat and there is no longer any teleological feature. Since Δ is null and $\mathcal{L}_\ell q_{ab} = 0$, the area of *any* of its cross sections is the same, denoted by a_Δ. As one would expect, one can show that there is no flux of gravitational radiation or matter across Δ. This captures the idea that the black hole itself is in equilibrium. Condition ii) is a rather weak 'energy condition' which is satisfied by all matter fields normally considered in classical general relativity. The non-trivial condition is i). It extracts from the notion of a Killing horizon just a 'tiny part' that refers only to the intrinsic geometry of Δ. As a result, every Killing horizon $\mathcal{K}$ is, in particular, an isolated horizon. However, a space-time with an isolated horizon Δ can admit gravitational radiation and dynamical matter fields away from Δ. In fact as a family of Robinson-Trautman space-times illustrates, gravitational radiation could even be present arbitrarily close to Δ. Because of these possibilities, there are many non-trivial examples and the transition from event horizons of stationary space-times to isolated horizons represents a significant generalization of black hole mechanics.[2]

An immediate consequence of the requirement $\mathcal{L}_\ell q_{ab} = 0$ is that there exists a 1-form ω_a on Δ such that $D_a \ell^b = \omega_a \ell^b$. Following the definition of κ on a Killing horizon, the *surface gravity* $\kappa_{(\ell)}$ of (Δ, ℓ) is defined as $\kappa_{(\ell)} = \omega_a \ell^a$. Again, under $\ell^a \to c\ell^a$, we have $\kappa_{(c\ell)} = c\kappa_\ell$. Together with Einstein's equations, the two conditions of Definition 1 imply $\mathcal{L}_\ell\, \omega_a = 0$ and $\ell^a D_{[a}\omega_{b]} = 0$. The Cartan identity relating the

[2]In fact the derivation of the zeroth and the first law requires slightly weaker assumptions, encoded in the notion of a 'weakly isolated horizon' (Ashtekar et al 2000, 2001).

Lie and exterior derivative now yields

$$D_a(\omega_b \ell^b) \equiv D_a \kappa_{(\ell)} = 0\,. \tag{3.6}$$

Thus, surface gravity is constant on every isolated horizon. This is the zeroth law, extended to horizons representing local equilibrium. In presence of an electromagnetic field, Definition 1 and the field equations imply: $\mathcal{L}_\ell F_{\underleftarrow{ab}} = 0$ and $\ell^a F_{\underleftarrow{ab}} = 0$. The first of these equations implies that one can always choose a gauge in which $\mathcal{L}_\ell A_{\underleftarrow{a}} = 0$. By Cartan identity it then follows that the electrostatic potential $\Phi_{(\ell)} := A_a \ell^a$ is constant on the horizon. This is the Maxwell analog of the zeroth law.

In this setting, the first law is derived using a Hamiltonian framework (Ashtekar et al 2000, 2001). For concreteness, let us assume that we are in the asymptotically flat situation and the only gauge field present is electromagnetic. One begins by restricting oneself to horizon geometries such that Δ admits a rotational vector field φ^a satisfying[3] $\mathcal{L}_\varphi q_{ab} = 0$. One then constructs a phase space $\mathbf{\Gamma}$ of gravitational and matter fields such that i) $\mathbb{M}$ admits an internal boundary Δ which is an isolated horizon; and ii) all fields satisfy asymptotically flat boundary conditions at infinity. Note that the horizon geometry is allowed to vary from one phase space point to another; the pair (q_{ab}, D) induced on Δ by the space-time metric only has to satisfy Definition 1 and the condition $\mathcal{L}_\varphi q_{ab} = 0$.

Let us begin with angular momentum. Fix a vector field ϕ^a on $\mathbb{M}$ which coincides with the fixed φ^a on Δ and is an asymptotic rotational symmetry at infinity. (Note that ϕ^a is not restricted in any way in the bulk.) Lie derivatives of gravitational and matter fields along ϕ^a define a vector field $\mathbf{X}(\phi)$ on $\mathbf{\Gamma}$. One shows that it is an infinitesimal canonical transformation, i.e., satisfies $\mathcal{L}_{\mathbf{X}(\phi)}\mathbf{\Omega} = 0$ where $\mathbf{\Omega}$ is the symplectic structure on $\mathbf{\Gamma}$. The Hamiltonian $\mathbf{H}(\phi)$ generating this canonical transformation is given by:

$$\mathbf{H}(\phi) = J^{(\phi)}_\Delta - J^{(\phi)}_\infty \quad \text{where} \quad J^{(\phi)}_\Delta = -\frac{1}{8\pi G}\oint_{\mathbb{S}} (\omega_a \phi^a)\,\epsilon - \frac{1}{4\pi}\oint_{\mathbb{S}} (A_a \phi^a)\,{}^\star F \tag{3.7}$$

where $J^{(\phi)}_\infty$ is the ADM angular momentum at infinity, $\mathbb{S}$ is any cross-section of Δ and ϵ the area element thereon. The term $J^{(\phi)}_\Delta$ is independent of the choice of $\mathbb{S}$ made in its evaluation and interpreted as the *horizon angular momentum.* It has numerous properties that support this interpretation. In particular, it yields the standard angular momentum expression in Kerr-Newman space-times.

To define horizon energy, one has to introduce a 'time-translation' vector field t^a. At infinity, t^a must tend to a unit time translation. On Δ, it must be a symmetry of q_{ab}. Since ℓ^a and φ^a are both horizon symmetries, $t^a = c\ell^a + \Omega\varphi^a$ on Δ, for some constants c and Ω. However, unlike ϕ^a, the restriction of t^a to Δ can not be fixed once and for all but must be allowed to vary from one phase space point to another.

[3] In fact for black hole mechanics, it suffices to assume only that $\mathcal{L}_\varphi \epsilon_{ab} = 0$ where ϵ_{ab} is the intrinsic area 2-form on Δ. The same is true on dynamical horizons discussed in the next section.

In particular, on physical grounds, one expects Ω to be zero at a phase space point representing a non-rotating black hole but non-zero at a point representing a rotating black hole. This freedom in the boundary value of t^a introduces a qualitatively new element. The vector field $\mathbf{X}(t)$ on $\mathbf{\Gamma}$ defined by the Lie derivatives of gravitational and matter fields does not, in general, satisfy $\mathcal{L}_{\mathbf{X}(t)}\,\mathbf{\Omega} = 0$; it need not be an infinitesimal canonical transformation. The necessary and sufficient condition is that $(\kappa_{(c\ell)}/8\pi G)\delta a_\Delta + \Omega\delta J_\Delta + \Phi_{(c\ell)}\delta Q_\Delta$ be an exact variation. That is, $\mathbf{X}(t)$ generates a Hamiltonian flow if and only if there exists a function $E^{(t)}_\Delta$ on $\mathbf{\Gamma}$ such that

$$\delta E^{(t)}_\Delta = \frac{\kappa_{(c\ell)}}{8\pi G}\delta a_\Delta + \Omega\delta J_\Delta + \Phi_{(c\ell)}\delta Q_\Delta \tag{3.8}$$

This is precisely the first law. Thus, the framework provides a deeper insight into the origin of the first law: It is the necessary and sufficient condition for the evolution generated by t^a to be Hamiltonian. (3.8) is a genuine restriction on the choice of phase space functions c and Ω, i.e., of restrictions to Δ of evolution fields t^a. It is easy to verify that $\mathbb{M}$ admits many such vector fields. Given one, the Hamiltonian $H(t)$ generating the time evolution along t^a takes the form

$$\mathbf{H}(t) \;=\; E^{(t)}_\infty - E^{(t)}_\Delta\,, \tag{3.9}$$

re-enforcing the interpretation of $E^{(t)}_\Delta$ as the horizon energy.

In general, there is a multitude of first laws, one for each vector field t^a, the evolution along which preserves the symplectic structure. In the Einstein-Maxwell theory, given *any* phase space point, one can choose a canonical boundary value t^a_o exploiting the uniqueness theorem. $E^{(t_o)}_\Delta$ is then called the horizon mass and denoted simply by m_Δ. In the Kerr-Newman family, $\mathbf{H}(t_o)$ vanishes and m_Δ coincides with the ADM mass m_∞. Similarly, if ϕ^a is chosen to be a global rotational Killing field, $J^{(\phi)}_\Delta$ equals $J^{(\phi)}_\infty$. However, in more general space-times where there is matter field or gravitational radiation outside Δ, these equalities do not hold; m_Δ and J_Δ represent quantities associated with the *horizon alone* while the ADM quantities represent the *total* mass and angular momentum in the space-time, including contributions from matter fields and gravitational radiation in the exterior region. In the first law (3.8), only the contributions associated with the horizon appear.

When the uniqueness theorem fails, as for example in the Einstein-Yang-Mills-Higgs theory, first laws continue to hold but the horizon mass m_Δ becomes ambiguous. Interestingly, these ambiguities can be exploited to relate properties of hairy black holes with those of the corresponding solitons.

4 Dynamical situations

A natural question now is whether there is an analog of the second law of thermodynamics. Using event horizons, Hawking showed that the answer is in the affirmative (see Hawking & Ellis 1973). Let $(\mathbb{M}, g_{ab})$ admit an event horizon $\mathcal{E}$. Denote by ℓ^a a geodesic null normal to $\mathcal{E}$. Its expansion is defined as $\theta_{(\ell)} := q^{ab}\nabla_a \ell_b$, where q^{ab} is any inverse of the degenerate intrinsic metric q_{ab} on $\mathcal{E}$, and determines the rate of change of the area-element of $\mathcal{E}$ along ℓ^a. Assuming that the null energy condition and Einstein's equations hold, the Raychaudhuri equation immediately implies that if $\theta_{(\ell)}$ were to become negative somewhere it would become infinite within a finite affine parameter. Hawking showed that, if there is a globally hyperbolic region containing $I^-(\mathcal{I}^+) \cup \mathcal{E}$ —i.e., if there are no naked singularities— this can not happen, whence $\theta(\ell) \geq 0$ on $\mathcal{E}$. Hence, if a cross-section S_2 of $\mathcal{E}$ is to the future of a cross section S_1, we must have $a_{S_2} \geq a_{S_1}$. Thus, in any (i.e., not necessarily infinitesimal) dynamical process, the change Δa in the horizon area is always non-negative. This result is known as the *second law of black hole mechanics.* As in the first law, the analog of entropy is the horizon area.

It is tempting to ask if there is a local physical process directly responsible for the growth of area. For event horizons, the answer is in the negative since they can grow in a flat porion of space-time. However, one can introduce quasi-local horizons also in the dynamical situations and obtain the desired result (Ashtekar & Krishnan 2003). These constructions are strongly motivated by earlier ideas introduced by Hayward (1994).

Definition 2: A 3-dimensional space-like sub-manifold $\mathcal{H}$ of $(\mathbb{M}, g_{ab})$ is said to be a *dynamical horizon* if it admits a foliation by compact 2-manifolds $\mathbb{S}$ (without boundary) such that:

i) The expansion $\theta_{(\ell)}$ of one (future directed) null normal field ℓ^a to $\mathbb{S}$ vanishes and the expansion of the other (future directed) null normal field, n^a is negative; and

ii) $-T^a{}_b \ell^b$ is a future pointing causal vector on $\mathcal{H}$.

One can show that this foliation of $\mathcal{H}$ is unique and that $\mathbb{S}$ is either a 2-sphere or, under degenerate and physically over restrictive conditions, a 2-torus. Each leaf $\mathbb{S}$ is a marginally trapped surface and referred to as a *cut* of $\mathcal{H}$. Unlike event horizons $\mathcal{E}$, dynamical horizons $\mathcal{H}$ are locally defined and do not display any teleological feature. In particular, they can not lie in a flat portion of space-time. Dynamical horizons commonly arise in numerical simulations of evolving black holes as world tubes of apparent horizons. As the black hole settles down, $\mathcal{H}$ asymptotes to an isolated horizon Δ , which tightly hugs the asymptotic future portion of the event horizon. However, during the dynamical phase, $\mathcal{H}$ typically lies well inside $\mathcal{E}$.

The two conditions in Definition 2 immediately imply that the area of cuts of $\mathcal{H}$ increases monotonically along the 'outward direction' defined by the projection of ℓ^a on $\mathcal{H}$. Furthermore, this change turns out to be directly related to the flux of energy falling across $\mathcal{H}$. Let R denote the 'radius function' on $\mathcal{H}$ so that the area of any cut $\mathbb{S}$ is given by $a_{\mathbb{S}} = 4\pi R^2$. Let N denote the norm of $\partial_a R$ and $\Delta\mathcal{H}$, the portion of $\mathcal{H}$ bounded by two cross-sections $\mathbb{S}_1$ and $\mathbb{S}_2$. The appropriate energy turns out to be associated with the vector field $N\ell^a$ where ℓ^a is normalized such that its projection on $\mathcal{H}$ is the unit normal $\hat{r}^a$ to the cuts $\mathbb{S}$. In the generic and physically interesting case when $\mathbb{S}$ is a 2-sphere, the Gauss and the Codazzi (i.e. constraint) equations imply:

$$\frac{1}{2G}\,(R_2 - R_1) = \int_{\Delta\mathcal{H}} T_{ab}\,N\ell^a\hat{\tau}^b d^3V + \frac{1}{16\pi G}\int_{\Delta\mathcal{H}} N\,\left(\sigma_{ab}\sigma^{ab} + 2\zeta_a\zeta^a\right)\,d^3V. \quad (4.10)$$

Here $\hat{\tau}^a$ is the unit normal to $\mathcal{H}$, σ^{ab}, is the shear of ℓ^a (i.e., the trace-free part of $q^{am}q^{bm}\nabla_m\ell_n$) and $\zeta^a = q^{ab}\hat{r}^c\nabla_c\ell_b$, where q^{ab} is the projector onto the tangent space of the cuts $\mathbb{S}$. The first integral on the right can be directly interpreted as the flux across $\Delta\mathcal{H}$ of matter-energy (relative to the vector field $N\ell^a$). The second term is purely geometric and is interpreted as the flux of energy carried by gravitational waves across $\Delta\mathcal{H}$. It has several properties which support this interpretation. Thus, not only does the second law of black hole mechanics hold for a dynamical horizon $\mathcal{H}$, but the 'cause' of the increase in the area can be directly traced to physical processes happening near $\mathcal{H}$.

Another natural question is whether the first law (3.8) can be generalized to fully dynamical situations, where δ is replaced by a finite transition. Again, the answer is in the affirmative. We will outline the idea for the case when there are no gauge fields on $\mathcal{H}$. As with isolated horizons, to have a well-defined notion of angular momentum, let us suppose that the intrinsic 3-metric on $\mathcal{H}$ admits a rotational Killing field φ. Then, the angular momentum associated with any cut $\mathbb{S}$ is given by

$$J_{\mathbb{S}}^{(\varphi)} = -\frac{1}{8\pi G}\oint_{\mathbb{S}} K_{ab}\varphi^a\hat{r}^b\,d^2V \;\equiv \frac{1}{8\pi G}\oint_{\mathbb{S}} j^{(\varphi)}d^2V\,, \quad (4.11)$$

where K_{ab} is the extrinsic curvature of $\mathcal{H}$ in $(\mathbb{M}, g_{ab})$ and $j^{(\varphi)}$ is interpreted as 'the angular momentum density'. Now, in the Kerr family, the mass, surface-gravity and the angular velocity can be unambiguously expressed as well-defined functions $\bar{m}(a,J)$, $\bar{\kappa}(a,J)$ and $\bar{\Omega}(a,J)$ of the horizon area a and angular momentum J. The idea is to use these expressions to associate mass, surface gravity and angular velocity with each cut of $\mathcal{H}$. Then, a surprising result is that the difference between the horizon masses associated with cuts $\mathbb{S}_1$ and $\mathbb{S}_2$ can be expressed as the integral of a *locally defined* flux across the portion $\Delta\mathcal{H}$ of $\mathcal{H}$ bounded by $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\begin{aligned}\bar{m}_2 - \bar{m}_1 \;&=\; \frac{1}{8\pi G}\int_{\Delta\mathcal{H}}\bar{\kappa}da \\ &+\; \frac{1}{8\pi G}\Big\{\oint_{\mathbb{S}_2}\bar{\Omega}j^{\varphi}\,d^2V - \oint_{\mathbb{S}_1}\bar{\Omega}j^{\varphi}\,d^2V - \int_{\bar{\Omega}_1}^{\bar{\Omega}_2} d\bar{\Omega}\oint_{\mathbb{S}} j^{\varphi}\,d^2V\Big\} \quad (4.12)\end{aligned}$$

If the cuts $\mathbb{S}_2$ and $\mathbb{S}_1$ are only infinitesimally separated, this expression reduces precisely to the standard first law involving infinitesimal variations. Therefore, (4.12) is *an integral generalization of the first law.*

Let us conclude with a general perspective. On the whole, in the passage from event horizons in stationary space-times to isolated horizons and then to dynamical horizons, one considers increasingly more realistic situations. In all three cases, the analysis has been extended to allow the presence of a cosmological constant Λ. (The only significant change is that the topology of cuts $\mathbb{S}$ of dynamical horizons is restricted to be $\mathbb{S}^2$ if $\Lambda > 0$ and is completely unrestricted if $\Lambda < 0$.) In the first two frameworks, results have also been extended to higher dimensions. Since the notions of isolated and dynamical horizons make no reference to infinity, these frameworks can be used also in spatially compact space-times. The notion of an event horizon, by contrast, does not naturally extend to these space-times. On the other hand, the generalization (2.4) of the first law (2.3) is applicable to event horizons of stationary space-times in a wide class of theories while so far the isolated and dynamical horizon frameworks are tied to general relativity (coupled to matter satisfying rather weak energy conditions). From a mathematical physics perspective, extension to more general theories is an important open problem.

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