Compare Analytical and Numerical Binary Coalescing Waveforms

Yi Pan
University of Maryland
In collaboration with A. Buonanno, J. Baker, J. Centrella, B. Kelly, S. McWilliams, F. Pretorius, and J. van Meter

IGC Inauguration Conference
Pennsylvania State University
August 2007
Motivation

- Comparable-mass binary black holes of stellar mass merge in LIGO band, with the highest signal-to-noise ratio.

- Numerical relativity (NR) gives full merger waveforms, but thousands of waveform templates may be needed to cover the binary parameter space – impossible demand for NR alone.

- To detect the most promising sources, we need to combine NR and post-Newtonian (PN) results to build full analytical waveforms.
Standard PN Waveforms

- Adiabatic PN waveforms in frequency domain
  - Restricted PN waveform \( h(t) = \omega(t)^{2/3} \cos \phi(t) \)
  - Frequency and phase evolution from energy balance equation
    \[
    \frac{d^2 \phi}{dt^2} = \frac{d\omega}{dt} = -\frac{\mathcal{F}}{dE/d\omega}
    \]
    - Frequency domain waveform from stationary-phase approximation
      \[
      \tilde{h}(f) \propto f^{-7/6} e^{i\psi(f)}
      \]
      \[
      \psi(f) = 2\pi ft_c - \phi_c - \pi/4 + \lambda_{\text{Newt}} f^{-5/3} + \lambda_{1\text{PN}} f^{-1} + \lambda_{1.5\text{PN}} f^{-2/3} \\
      + \lambda_{2\text{PN}} f^{-1/3} + \ldots + \lambda_{3.5\text{PN}} f^{2/3} + \lambda_{4\text{PN}} \log f
      \]

- Waveform closeness estimated through the fitting factor (FF):
  \[
  \langle s, t \rangle \equiv 4Re \int_0^\infty \frac{\tilde{s}(f)\tilde{t}^*(f)}{S_n(f)} df \\
  FF = \max_{t_0, \phi_0, \ldots} \frac{\langle s, t \rangle}{\sqrt{\langle s, s \rangle \langle t, t \rangle}}
  \]
Standard PN waveforms for GW Detection

- Standard PN waveforms at 3.5PN order work rather well until the innermost stable circular orbit
- To match plunge-merger, we introduce a pseudo 4PN term
  - The 4PN term $\lambda_{4PN} \log f$ corrects phase evolution
  - Cutoff frequency at final BH quasi-normal-mode frequency gives “ring-down”

amplitudes normalized
FF > 0.97
Black Hole Quasi-Normal Modes

- Black hole quasi-normal modes (QNMs):
  [Vishveshwara 70; Press 71; Chandrasekhar & Detweiler 75; Schutz & Will 85]
  \[-\frac{\partial^2}{\partial r^2} \psi + V \psi = \omega^2_{QNM} \psi \quad \omega_{QNM}(m, a)\]
  In the case of Schwarzschild, twice the light-ring (orbital) frequency close to the fundamental QNM frequency

- Close limit approximation
  [Price & Pullin 94]
  Switching from a two-body to a one-body description and use BH perturbation theory to describe the subsequent evolution
Effective-One-Body (EOB) Model

[Buonanno and Damour, 99]

- Improve PN convergence during the last stages of inspiral and plunge by re-summing the PN dynamics
- Mapping the real conservative two-body dynamics onto an effective one-body problem: a test particle moving in some effective background metric
  - Mapping obtained in the Hamilton-Jacobi formalism by imposing that the real and effective adiabatic invariants coincide:
    \[ M_0 = M = m_1 + m_2 ; \quad m_0 = \mu = m_1 m_2 / M ; \]
    \[ J_{\text{eff}} = J_{\text{real}} ; \quad I_{\text{eff}} = I_{\text{real}} ; \]
  - In non-spinning case, background metric is chosen to be \( \nu \)-deformed Schwarzschild:
    \[ ds_{\text{eff}}^2 = -A_{\nu}(r)c^2 dt^2 + \frac{D_{\nu}(r)}{A_{\nu}(r)} dr^2 + r^2 d\Omega^2 \]
- All conservative dynamics condensed in two coefficients \( A_{\nu}(r) \) and \( D_{\nu}(r) \)
- Radiation-reaction effect is included as a force in the Hamilton formalism
• Real Hamiltonian:

\[
\mathcal{H}_\text{real}(Q, P) = \sqrt{1 + 2\nu \mathcal{H}_\text{eff}(q, p) - 1}
\]

\[
\mathcal{H}_\text{eff}(q, p) = \sqrt{A_\nu(q) \left[ 1 + p^2 + \left( \frac{A_\nu(q)}{D_\nu(q)} - 1 \right) (n \cdot p)^2 + T_4(p) \right]}
\]

• \(A_\nu(q)\) and \(D_\nu(q)\) at 3.5PN

\[
A^{3\text{PN}}(q) = 1 - 2/q + 2\nu/q^3 + 18.7\nu/q^4
\]

\[
D^{3\text{PN}}(q) = 1 - 6\nu/q^2 + 2\nu(3\nu - 26)/q^3
\]

• EOB beyond inspiral [Buonanno and Damour, 00]
  – The plunge is a smooth continuation of the adiabatic inspiral phase
  – The transition from merger to ring-down was assumed very short
  – 3 QNMs are attached imposing the continuity of the waveforms and its higher order time derivatives at the EOB light-ring position
Compare NR and EOB at 3.5PN

- For equal-mass binaries there is ONE dominant frequency
- Frequency increases fast during merger
- Mass and spin of the final BH computed by combining light-ring information with loss of energy and angular momentum during ring-down

[Buonanno, Cook, and Pretorius]
Improved EOB model

- Modify the EOB radial potential $A_y(r)$
  \[ \omega_{\text{plunge}} = \frac{A(r) \ p_\phi}{r^2 \ H^2} \]
  \[ A^{p4PN}(r) = A^{3PN}(r) + \lambda \nu / r^5 \quad (\lambda = 60) \]
  - Apply Pade re-summation to ensure presence of LSO and light ring
  - p4PN term moves LR & LSO inward

- Final BH mass and spin from NR
  - Mass of final BH
    \[ M_{\text{BH}} / m = 1 + (\sqrt{8/9} - 1)\nu - 0.498\nu^2 \]
  - Spin of final BH
    \[ a_{\text{BH}} / M_{\text{BH}} = \sqrt{12}\nu - 2.90\nu^2 \]
  - QNM frequency given by $M_{\text{BH}}$ and $a_{\text{BH}}$
Compare NR and Improved EOB: Equal-Mass

Equal-mass case: phase difference ~6% of a GW cycle, FF>0.98
Compare NR and Improved EOB: Unequal-Mass I

Mass ratio 4:1, consider the first 4 multipole modes
Compare NR and Improved EOB: Unequal-Mass II

Mass ratio 4:1, FF>0.98.
Although there is problem matching the ring-down of $h_{44}$, the effect on full waveform is negligible

![Graph of NR and EOB waveforms for unequal mass systems.](image-url)
Conclusions

• The joined work of the NR and PN communities is producing very accurate analytical waveforms for gravitational-wave detection and parameter estimation.

• “Simplicity” of the NR waveforms for non-spinning comparable mass BHs moving on quasi-circular orbits: single dominant frequency and QNMs triggered by resonance during the fast rise of the frequency.

• For known PN results, 3.5PN waveforms match NR results the best.

• Extend the EOB improvements to spinning, precessing binaries.